A MATHEMATICAL NOTE

Upper bound on the number of N-spheres that can simultaneously kiss a central sphere

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Introduction. I take my inspiration from an article in this week's issue of *Science News* (Vol. 166, No. 14) that summarized recent progress toward solution of the SPHERE PACKING and SPHERE KISSING problems in N-dimensions. It is some very simple aspects of the kissing problem that will concern me here.

The maximal number of unit spheres that can simultaneously kiss a central unit sphere is a dimension-dependent integer—call it $k(N)$. Trivially $k(1) = 2$, while by a famous construction (FIGURE 1) $k(2) = 6$. The description of $k(N)$

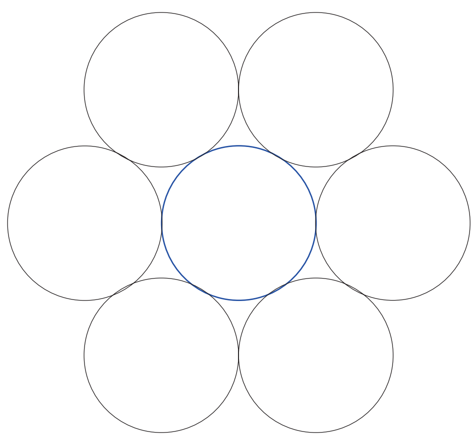

FIGURE 1: *This simplest instance of the kissing problem is solved by direct construction, which gives* $k(2) = 6$.

is—surprisingly—a famously difficult problem. No formula exists that supplies $k(N)$ in the general case: at present, each case must be approached individually. My objective is, by elementary means, to supply an *upper bound* on $k(N)$.

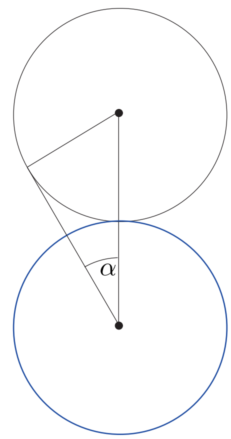

FIGURE 2: *Figure used to demonstrate that a kissing circle subtends a central angle of* $60°$.

1. Analytical aspects of the case N = 2. We ask: *What central angle σ is subtended by a kissing circle*? From FIGURE 2 is becomes obvious that

$$\sigma = 2\alpha \quad \text{with} \quad \alpha = \arcsin\tfrac{1}{2} = \tfrac{1}{6}\pi \tag{1}$$

which supplies

$$\sigma = \frac{2\pi}{6}$$

and from this information we recover $k(2) = 6$.

2. The case N = 3. The first thing to notice about FIGURE 3 is that in cross section it reproduces FIGURE 2. Our problem therefore is to compute the area of a spherical cap with the same central semi-angle α as before. The calculation (see FIGURE 4) is elementary

$$\begin{aligned} \text{spherical cap area } C_3(\alpha; r) &= \int_0^\alpha 2\pi r \sin\theta \cdot r\, d\theta \\ &= 2\pi r^2(1 - \cos\alpha) \end{aligned} \tag{2.1}$$

As a check, we have

$$\text{total surface area } S_3(r) = C_3(\pi; r) = 4\pi r^2 \tag{2.2}$$

whence

$$\text{volume } V_3(r) = \int_0^r S_3(\rho)\, d\rho = \tfrac{4}{3}\pi r^3 \tag{2.3}$$

The relevant implication is that

$$\frac{S_3(r)}{C_3(\frac{\pi}{6}; r)} = \frac{4}{2 - \sqrt{3}} = 14.9282$$

from which we conclude that

$$k(3) \leqslant 14 \tag{3}$$

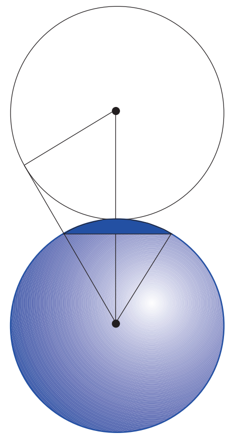

FIGURE 3: *In three dimensions the problem is to discover what fraction of the spherical surface is taken up by the cap.*

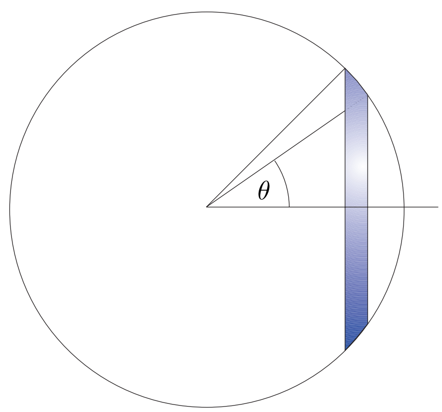

FIGURE 4: *Figure used to compute the area of a spherical cap in the 3-dimensional case.*

3. Hyperspherical essentials. Standardly, one writes

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dxdy = \left[\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right]^2 &= \int_0^\infty e^{-r^2} 2\pi r\,dr \\ &= \pi\int_0^\infty e^{-u}\,du \\ &= \pi\end{aligned}$$

to arrive at the statement

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

basic to the theory of Gaussian integration. Standing that procedure on its head, we have

$$\begin{aligned}\int\cdots\int_{-\infty}^{+\infty} e^{-(x_1^2+x_2^2+\cdots+x_N^2)}\,dx_1dx_2\cdots dx_N &= \int_0^\infty e^{-r^2}\,\mathcal{S}_N r^{N-1}\,dr\\ &= \mathcal{S}_N\cdot\tfrac{1}{2}\Gamma\big(\tfrac{N}{2}\big)\end{aligned}$$

giving

$$\begin{aligned}S_N(r) &= \mathcal{S}_N\cdot r^{N-1}\\ &= \frac{\sqrt{\pi^N}}{\frac{1}{2}\Gamma\big(\frac{N}{2}\big)}\,r^{N-1} \qquad (4.1)\\ &= \text{surface area of an } N\text{-sphere}\end{aligned}$$

and

$$\begin{aligned}V_N(r) &= \frac{\sqrt{\pi^N}}{N\cdot\frac{1}{2}\Gamma\big(\frac{N}{2}\big)}\,r^{N} \qquad (4.2)\\ &= \text{volume of an } N\text{-sphere}\end{aligned}$$

4. The general case. Proceeding in mimicry of (2.1), which can be written

$$C_3(\alpha;r) = \int_0^\alpha S_2(r\sin\theta)\cdot r d\theta$$

we expect to have

$$C_N(\alpha;r) = \int_0^\alpha S_{N-1}(r\sin\theta)\cdot r d\theta \qquad (5)$$

As a check we look to some low-order cases

$$\begin{aligned}C_2(\alpha;r) &= 2r\alpha\\ C_3(\alpha;r) &= 2\pi r^2(1-\cos\alpha)\\ C_4(\alpha;r) &= 4\pi r^3(\tfrac{1}{2}\alpha-\tfrac{1}{4}\sin 2\alpha)\\ C_5(\alpha;r) &= 2\pi^2 r^4(\tfrac{2}{3}-\tfrac{3}{4}\cos\alpha+\tfrac{1}{12}\cos 3\alpha)\\ C_6(\alpha;r) &= 8\pi^2 r^5(\tfrac{1}{8}\alpha-\tfrac{1}{12}\cos 2\alpha+\tfrac{1}{96}\cos 4\alpha)\\ &\vdots\end{aligned}$$

and verify that in all those cases (and all others that I have asked *Mathematica* to check)

$$C_N(\pi,r) = S_N(r)$$

which is to say: full cap = entire surface.

We are placed thus in position to assert that

$$k(N) \leqslant \frac{S_N(r)}{C_N(\frac{\pi}{6},r)} \qquad (6)$$

from which the r factors cancel, and on which basis I have constructed the following table:

$$
\begin{aligned}
k(2) &\leqslant 6\\
k(3) &\leqslant 14\\
k(4) &\leqslant 34\\
k(5) &\leqslant 77\\
k(6) &\leqslant 170\\
k(7) &\leqslant 368\\
k(8) &\leqslant 788\\
k(9) &\leqslant 1673\\
k(10) &\leqslant 3527\\
&\ \vdots\\
k(24) &\leqslant 89{,}437{,}026\\
&\ \vdots
\end{aligned}
$$

It seems a little surprising that so much kissing goes (or *could* go) on in high dimension, since the hypervolume $V_N(1)$ (whence also the hypersurface area) is well known to approach zero as $N\uparrow\infty$:

$$
\begin{aligned}
V_2(1) &= 3.1415\\
V_3(1) &= 4.1888\\
V_4(1) &= 4.9348\\
V_5(1) &= 5.2638\\
V_6(1) &= 5.1677\\
V_7(1) &= 4.7248\\
V_8(1) &= 4.0587\\
V_9(1) &= 3.2985\\
V_{10}(1) &= 2.5502\\
&\ \vdots\\
V_{24}(1) &= 0.0019\\
&\ \vdots
\end{aligned}
$$

5. Asymptotic approximation. At (6) we achieved a result that can be spelled out

$$
\begin{aligned}
k(N) &\leqslant \frac{\sqrt{\pi^N}}{\frac{1}{2}\Gamma\big(\frac{N}{2}\big)}\cdot\left[\frac{\sqrt{\pi^{N-1}}}{\frac{1}{2}\Gamma\big(\frac{N-1}{2}\big)}\int_0^{\frac{1}{6}\pi}\sin^{N-2}\theta\,d\theta\right]^{-1}\\
&= \sqrt{\pi}\;\frac{\Gamma\big(\frac{N-1}{2}\big)/\Gamma\big(\frac{N}{2}\big)}{\frac{1}{(N-1)2^{N-1}}\,\texttt{Hypergeometric2F1}\big[\,\frac{1}{2},\frac{N-1}{2},\frac{N+1}{2},\frac{1}{4}\,\big]}
\end{aligned}
$$

In Stirling approximation

$$\Gamma(z) \sim \sqrt{2\pi/z}\;(z/e)^z$$

which gives

$$\begin{aligned}\sqrt{\pi}(N-1)2^{N-1}\Gamma\left(\tfrac{N-1}{2}\right)/\Gamma\left(\tfrac{N}{2}\right) &\sim 2^N\sqrt{\tfrac{1}{2}N\pi e\left(1-\tfrac{1}{N}\right)^N}\\ &\sim 2^N\sqrt{\tfrac{1}{2}N\pi}\end{aligned}$$

Discussion of the asymptotics of $k(N)$ hinges, therefore, on the asymptotics of ${}_2F_1(a,b;c;z)$, but this is an intricate business[1] into which—for a reason that will soon emerge—I have no strong motivation to enter.

6. Discussion. My results are certainly *not* new, and are of little interest in themselves, but acquire some interest for the surprising light that they cast on what is presently known about the kissing problem.

Fairly recent scholarship has dredged from the papers of David Gregory (1659–1708) the information that it was on 4 May 1694 that Gregory and Isaac Newton (1642–1725), while considering an astronomical problem, fell into a discussion of $k(3)$. Newton asserted—on what grounds? Had he looked into the matter experimentally? Or was he simply repeating an assertion made by Kepler in his *Six-cornered Snow*?—that $k(3) = 12$, while Gregory was of the opinion that perhaps $k(3) = 13$ (which itself seems pretty remarkable, since our $k(3) \leqslant 14$ leads one to ask how Gregory came to be smart enough to be so conservative). The point at issue came to be known as "Newton's thirteen spheres problem" (why thirteen? Perhaps Newton counted the central sphere, on grounds that the kissee should be reckoned among the kissers: to be kissed is to kiss).

Not until 1874 was the value of $k(3)$ definitely established. And not until the early 1950s did the kissing problem begin to attract lively attention—an attention that was invigorated by the realization soon thereafter that the problem had direct relevance to the <u>design of codes</u> for the efficient tranmission and storage of information, and to the <u>numerical evaluation of N-dimensional integrals</u>. Upper bounds much more efficient than mine were devised by H. S. M. Coxeter in 1963 and by P. Delsarte in 1972: both supply $k(3) \leqslant 13$.

By the 1980s it had been established that

$$\begin{aligned} k(3) &= 12 &&: \quad \text{compare my} \quad k(3) \leqslant 14\\ k(4) &= 24 &&: \quad \text{compare my} \quad k(4) \leqslant 34\\ k(8) &= 240 &&: \quad \text{compare my} \quad k(8) \leqslant 788\\ k(24) &= 196{,}560 &&: \quad \text{compare my} \; k(24) \leqslant 89{,}437{,}026 \end{aligned}$$

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[1] See the reference to G. N. Watson's bessel function monograph that appears on page 11 of W. Magnus & F. Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1954).

What strikes me as remarkable is not that my theoretical upper bounds so grossly over-estimate the facts of the matter but that the facts fall so far short of my estimates: a lot of space is wasted in high dimension, the population of kissers becomes loose and progressively more loose as N becomes large. The kissing problem appears to acquire its difficulty from the circumstance that its solutions are devoid of crystaline regularity, of pattern: kissing—in more than two dimensions—is a sloppy business!

Google's "sphere kissing problem" search produces a great many hits.[2] For an engaging introduction to the problem, see

http://plus.maths.org/issue23/features/kissing/

A definitive source is J. H. Conway & N. J. Sloane, *Sphere Packings, Lattices & Groups* (3^{rd} edition 2004), which provides authoritative accounts not only of the mathematics but also of its applications...to coding theory, to quantum mehcanics, to string theory, *etc*.

Specialists in the problem report that some N-values yield much more easily to analysis than others, and that the case $N = 24$ is in many respects "magical." My own methods are much too crude to provide any hint of that fact, though they do expose the sense in which the case $N = 2$ is special.

ADDENDUM: Within brief hours of the time I distributed a copy of this material to Richard Crandall he responded, in reference to the discussion in §5, that asymptotically

$$f(N) \equiv {}_2F_1\left(\tfrac{1}{2}, \tfrac{N-1}{2}, \tfrac{N+1}{2}, \tfrac{1}{4}\right) \sim \sqrt{\tfrac{4}{3}}$$

Mathematica supplies $f(0) = \sqrt{\frac{3}{4}}$, and is happy to provide figures that are

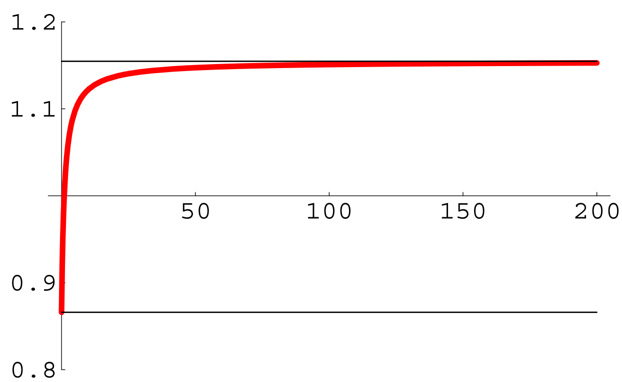

FIGURE 5: *Graph of* ${}_2F_1\left(\frac{1}{2}, \frac{N-1}{2}, \frac{N+1}{2}, \frac{1}{4}\right)$. *The horizontal lines mark the values of* $\sqrt{3/4}$ *and of* $\sqrt{4/3}$.

[2] "Kissing problem" leads, on the other hand, to a population of hits that is nearly 38 times larger, but relates to quite another issue!

consistent with Richard's assertion, but refuses to assign a value to $f(\infty)$ or to develop $f(N)$ in powers of N^{-1}. I asked Richard how, under those circumstances, he had proceeded. His methods, as he described them to me, are too pretty to be allowed simply to evaporate, so I record here an account of them.

By definition (Abramowitz & Stegun, **15.1.1**)

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} $$

where

$$ \begin{aligned} (a)_0 &\equiv 1 \\ (a)_n &\equiv \texttt{Pochhammer[a,n]} = a(a+1)(a+2)\cdots(a+n-1) \end{aligned} $$

Truncate the series to construct (say)[3]

$$ {}_2\mathcal{F}_1(a,b;c;z) \equiv \sum_{n=0}^{40} \frac{\texttt{Pochhammer[a,n]Pochhammer[b,n]}}{\texttt{Pochhammer[c,n]}}\frac{z^n}{n!} $$

and define

$$ \mathrm{f}(N) \equiv {}_2\mathcal{F}_1\big(\tfrac{1}{2}, \tfrac{N-1}{2}, \tfrac{N+1}{2}, \tfrac{1}{4}\big) $$

whence

$$ \mathrm{g}(N) \equiv \mathrm{f}(N^{-1}) $$

Leaving $\mathrm{g}(N)$—which would take *many* (!!) pages to write out—to reside within the computer's memory, we command `Series[g[n], {n,0,3}]` and obtain

$$ \mathrm{g}(n) = G_0 - G_1 n + G_2 n^2 - G_3 n^3 + \cdots $$

or again

$$ \mathrm{f}(N) = G_0 - G_1 N^{-1} + G_2 N^{-2} - G_3 N^{-3} + \cdots $$

where

$$ G_0 = \frac{421899181867037803574013790176444481874464786317}{365375409332725729550921208179070754913983135744} $$

and where the higher-order Gs are similarly preposterous ratios. What to do with such results? Proceeding on the hunch that they are trying to tell us

[3] It is my experience that *Mathematica* becomes confused if one at this point attempts to use

$$ (a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} $$

something simple, Richard commands `N[`G_0`,30]` and obtains

$$G_0 = 1.15470053837925152901829753687$$

which is not simple. But (he has the genius to notice)

$$\begin{aligned}(G_0)^2 &= 1.33333333333333333333333327761\\ &\approx \tfrac{4}{3}\end{aligned}$$

which *is* tellingly simple. Similarly

$$\begin{aligned}(G_1)^2 &= 0.148148148148148148148148148214661305\\ (G_2)^2 &= 0.592592592592592592592592592592234182761\end{aligned}$$

where again the repeating decimals suggest rational numbers. Indeed, from

$$1000(G_1)^2 - (G_1)^2 = 147.999999999999999999999984664$$

we find $(G_1)^2 \approx 148/999 = 4/27$, and similarly $(G_2)^2 \approx 592/999 = 16/27$.[4] The *strong implication* therefore is that

$$f(N) \sim \sqrt{\tfrac{4}{3}} - N^{-1}\sqrt{\tfrac{4}{27}} + N^{-2}\sqrt{\tfrac{16}{27}} - \cdots$$

Richard observes finally that an appeal to (see Abramowitz & Stegun **15.3.4**) the identity

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\cdot {}_2F_1\big(a,c-b;c;\tfrac{z}{z-1}\big)$$

permits one to write

$$\begin{aligned}f(N) &= \sqrt{\tfrac{4}{3}}\cdot {}_2F_1(\tfrac{1}{2},1;\tfrac{N+1}{2};-\tfrac{1}{3})\\ &= \sqrt{\tfrac{4}{3}}\cdot\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n(1)_n}{(\frac{N+1}{2})_n}\frac{(-\frac{1}{3})^n}{n!}\\ &= \sqrt{\tfrac{4}{3}}\cdot\Big\{1 - \frac{1}{3(N+1)} + \frac{1}{6(N+1)(1+\frac{N+1}{2})}\\ &\qquad\qquad - \frac{5}{36(N+1)(1+\frac{N+1}{2})(2+\frac{N+1}{2})} + \cdots\Big\}\end{aligned}$$

from which it becomes clear that in the limit $N\uparrow\infty$ one has

$$f(\infty) = \sqrt{\tfrac{4}{3}} \quad \textbf{exactly}$$

[4] I am indebted to Darrell Schroeter for reminding me of the method by which repeating decimals are most efficiently rendered as ratios. Curiously, Richard's procedure apparently *fails to yield rational numbers in next higher order*, though possibly (doubtfully) it would do so if all prior calculations had been carried out to higher order and with higher precision.